

# Spin-measurement retrodiction revisited

Steffen Metzger\*

*Lady Margaret Hall, University of Oxford, Oxford OX2 6QA, United Kingdom*

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## Abstract

The retrodiction of spin measurements along a set of different axes is revisited in detail. The problem is presented in two different pictures, a geometric and a general algebraic one. Explicit measurement operators that allow the retrodiction are given for the case of three and four axes. For the Vaidman-Aharanov-Albert case of three orthogonal axes the quantum network is constructed for two different initial Bell states.

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\*steffen.metzger@physik.uni-muenchen.de

# 1 Introduction

In contrast to classical mechanics, quantum mechanics does not allow all observables of a physical system to be measured simultaneously. Only for a set of mutually commuting observables definite values can be attributed to a physical system. Therefore it is one of the predictions of quantum mechanics that two spin components of a spin- $\frac{1}{2}$  particle cannot be measured simultaneously. That is, a given spin state cannot be an eigenstate of two or more spin operators  $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_l$ , with the usual Pauli matrices  $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$  and unit vectors  $\mathbf{n}_l$ . The result of a future measurement on a state can only be known in advance if the state is an eigenstate of the measurement operator. Now, if the spin is to be measured along a set of axes  $\{\mathbf{n}_l\}$ , then at most the result of one of the measurements can be predicted, namely when the spin state is an eigenstate of  $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_i$  for a specific  $i$ .

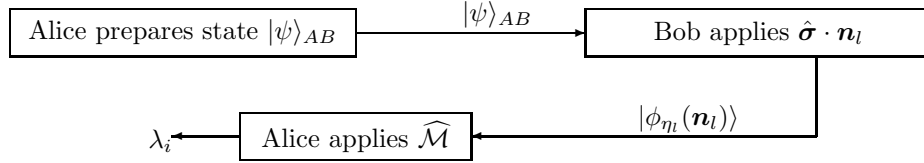
However, we can ask the question, whether it is possible to *retrodict* the results of a spin measurement along more than one possible axis. To be more specific we have the following problem: Alice prepares an initial spin- $\frac{1}{2}$  particle and gives it to Bob. Bob now performs one spin measurement along one of a set of axes the directions of which are known to Alice. However, Alice does not know which axis out of the set Bob chooses. After his measurement Bob gives the spin- $\frac{1}{2}$  particle back to Alice. Now Alice can perform another measurement on the particle, which should enable her to retrodict the value Bob obtained in his measurement. The fact that Alice does not yet know the axis along which Bob actually measures is most important. It means that she must be able to infer all possible results as a function of the axis that was chosen by Bob. After her measurement Bob tells her along which axis he actually measured and she has to tell him the result he got with certainty. This means that Bob can even cheat and tell Alice that he measured along axis 1 and Alice tells him which result (she believes!) he got. But then Bob confesses that he had lied and in fact measured along axis 2 and again Alice should be able to tell Bob the result he obtained if indeed he had measured along axis 2. And now Bob might have lied again and measured along another axis, and so on. This means that the measurement Alice makes has to extract information about all possible measurement axes out of the system, even though only one measurement was performed. The question now is: how should Alice prepare the initial spin- $\frac{1}{2}$  particle and what measurement does she have to perform on the particle Bob gives back to her?

To state the problem more clearly we put it into mathematical language. Alice prepares a spin- $\frac{1}{2}$  particle in a (possibly entangled) state  $|\psi\rangle_{AB} \in \mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$ .  $\mathcal{H}_B$  is the two-dimensional Hilbert space of the spin- $\frac{1}{2}$  particle on which Bob can perform a spin measurement and  $\mathcal{H}_A$  is an arbitrary Hilbert space. The state  $|\psi\rangle_{AB}$  is measured by Bob along an axis  $\mathbf{n}_l \in \{\mathbf{n}_l : 1 \leq l \leq m\}$ , i.e. Bob applies the measurement operator  $\mathbf{1} \otimes (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_l)$ . This measurement projects the original state onto an eigenstate

$$|\phi_{\eta_l}(\mathbf{n}_l)\rangle = \frac{1}{2}(\mathbf{1} \otimes \mathbf{1} + \eta_l \mathbf{1} \otimes (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_l))|\psi\rangle_{AB} \quad (1)$$

of the spin operator with eigenvalue  $\eta_l = \pm 1$ . Note that these states are not normalized if we assume normalisation for the initial state  $|\psi\rangle_{AB}$ . Now Alice can perform a measurement on  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$  and afterwards has to know the result  $\eta_l$  Bob obtained, along whatever axis he measured. We denote the operator Alice applies to project onto a basis by  $\widehat{\mathcal{M}} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$  with  $\{|\phi_j\rangle\}$  a basis of  $\mathcal{H}_{AB}$ . The task is to find  $|\psi\rangle_{AB}$  and  $\widehat{\mathcal{M}}$ .

The whole process can be visualized like this<sup>1</sup>:



Let us illustrate the problem by looking at the simple examples in which  $m=1$  and  $m=2$ , where  $|\psi\rangle_{AB}$  is unentangled and  $\mathcal{H}_A$  is trivial. Therefore  $\mathbf{1} \otimes (\hat{\sigma} \cdot \mathbf{n}_l)$  reduces to  $\hat{\sigma} \cdot \mathbf{n}_l$ . The  $m=1$  case is trivially solved by either Alice preparing  $|\psi\rangle_{AB}$  as an eigenstate of  $\hat{\sigma} \cdot \mathbf{n}_1$  or by Alice measuring along  $\mathbf{n}_1$  afterwards. In the first case the result is not even retrodicted but can be predicted.

The  $m=2$  can be solved easily as well. If Alice prepares an eigenstate of  $\hat{\sigma} \cdot \mathbf{n}_1$ , she knows in advance which result Bob gets if he applies  $\hat{\sigma} \cdot \mathbf{n}_1$ . To infer the value Bob obtains if he measures along  $\mathbf{n}_2$ , Alice applies the measurement operator  $\hat{\sigma} \cdot \mathbf{n}_2$  on the state Bob gave back to her. This procedure enables her to tell the value Bob obtained along whatever axis he measured. In fact, here we have a mixture of a prediction and a retrodiction, as the result of one possible measurement is known from the beginning.

At first glance the knowledge of spin components along different axes seems to contradict quantum mechanics. This led Vaidman, Aharonov and Albert (VAA) who first stated and solved the problem for three orthogonal axes to use the provocative title “How to Ascertain the Values of  $\hat{\sigma}_x$ ,  $\hat{\sigma}_y$  and  $\hat{\sigma}_z$  of a Spin- $\frac{1}{2}$  Particle” [1]. But a closer examination of the problem reveals that the kind of information Alice has about different spin components is not the one forbidden by quantum mechanics. Alice, of course, does not really know two or more eigenvalues of non-commuting operators but only extracts conditional information. The values Alice gets are a function of Bob’s measurement axis and not until she knows along which axis Bob actually measured do these values have any physical meaning. Only when the additional information about the choice of axis is revealed, *one* value is singled out. This one now does have physical significance, namely it is the one eigenvalue Bob actually obtained. This also points out that if Bob cheats and tells Alice that he had measured along an axis different from the one he actually used, the result Alice tells him is meaningless.

Obviously, the Hilbert space  $\mathcal{H}_A$  was not used in the  $m=1,2$  examples given above and it was sufficient to prepare an unentangled state in the two-dimensional space  $\mathcal{H}_B$ . A state in a two-dimensional Hilbert space is usually called a *qubit* and is written as  $a|\uparrow\rangle + b|\downarrow\rangle$ , where  $|\uparrow\rangle$  and  $|\downarrow\rangle$  denote an orthonormal basis. (For spin- $\frac{1}{2}$  particles these states are often eigenstates of spin measurements along a specific axis. E.g. for the z-axis they are denoted by  $|\uparrow\rangle_z$  and  $|\downarrow\rangle_z$ .)

The problem no longer has a simple solution if Bob can measure along three or more axes, which may or may not be orthogonal. Let us look at the case where Bob can measure along the three orthogonal axes  $x$ ,  $y$  and  $z$ , i.e. one of the measurement operators  $\mathbf{1} \otimes \hat{\sigma}_x$ ,  $\mathbf{1} \otimes \hat{\sigma}_y$ ,  $\mathbf{1} \otimes \hat{\sigma}_z$  is applied. We can convince ourselves that it is no longer sufficient to prepare an unentangled qubit if  $m \geq 3$ . Hence, if the problem can be solved at all,  $\mathcal{H}_A$  can no longer be trivial.

The problem as stated above was first presented by VAA in [1], who solved it for the  $m=3$  orthogonal case. Alice’s initial state was prepared as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_z \otimes |\uparrow\rangle_z + |\downarrow\rangle_z \otimes |\downarrow\rangle_z) \quad (2)$$

<sup>1</sup>The fact that Alice ‘sends’ her state to Bob should not be taken literally. She can as well prepare a state, leave and then Bob comes and looks at the state. Then spin states do not have to be parallelly transported and thus complications can be avoided.

and Alice's measurement operator  $\widehat{\mathcal{M}} = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|$  with  $|\phi_j\rangle$  as given in the appendix. In the next section a similar solution for the case that  $|\psi\rangle_{AB}$  is the singlet state will be derived.

In [2] Ben-Menahem used a more algebraic way to generalise the problem and showed that there are solutions for the cases  $m=3$  and  $m=4$ , but not for  $m \geq 5$ . However for  $m=4$ , the four axes Bob may measure along are no longer independent but have to satisfy the condition  $\sum_{l=1}^4 \mathbf{n}_l = 0$  for there to be a solution. In this case, the dimension of the Hilbertspace  $\mathcal{H}_{AB}$  is shown to be six. This space, however, is not sufficient to solve the  $m=3$  non-orthogonal case, where  $\mathcal{H}_{AB}$  has to be eight-dimensional.

Very recently, this method of spin-measurement retrodiction was used to set up a secure key distribution in quantum cryptography [3].

## 2 Solution for three orthogonal axes

After having stated the problem and given some simple examples with  $m=1$  and  $m=2$ , we can now try to construct a solution for higher dimensional cases. For the moment we restrict ourselves to the case in which Bob can measure along three orthogonal axes, that is either of the measurement operators  $\mathbf{1} \otimes \hat{\sigma}_x, \mathbf{1} \otimes \hat{\sigma}_y, \mathbf{1} \otimes \hat{\sigma}_z$  can be applied to the original state  $|\psi\rangle_{AB}$ . Each of the measurements has two possible outcomes  $\eta_l = \pm 1$ , so there are six possible results altogether, which are denoted by  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle \in \{|\uparrow, x\rangle, |\downarrow, x\rangle, |\uparrow, y\rangle, |\downarrow, y\rangle, |\uparrow, z\rangle, |\downarrow, z\rangle\} \subset \mathcal{H}_{AB}$ . Note that these are different from the basis states of the two-dimensional Hilbert space  $\mathcal{H}_B$  which are denoted by  $\{|\uparrow\rangle_x, |\downarrow\rangle_x\}, \{|\uparrow\rangle_y, |\downarrow\rangle_y\}$  or  $\{|\uparrow\rangle_z, |\downarrow\rangle_z\}$ . They will only coincide if the original state  $|\psi\rangle_{AB}$  is an unentangled qubit.<sup>2</sup>

Now Alice measures one of these states  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$ , i.e. she applies the operator  $\widehat{\mathcal{M}}$  to it and therefore projects the state onto a basis  $\{|\phi_j\rangle\}$  of  $\mathcal{H}_{AB}$  with distinct eigenvalues  $\lambda_j$  respectively. To ensure that her measurement gives the result she wants, Alice has to construct a look-up table. This table must tell her that if she measures e.g.  $\lambda_2$  Bob got 'up' if he measured along x, 'up' if he measured along y and 'down' if he measured along z. That is, Alice needs a table like this:

|             | x | y | z |
|-------------|---|---|---|
| $\lambda_1$ | ↓ | ↓ | ↓ |
| $\lambda_2$ | ↑ | ↑ | ↓ |
| $\lambda_3$ | ↓ | ↑ | ↑ |
| $\lambda_4$ | ↑ | ↓ | ↑ |

So if she measures  $\lambda_2$ , for example, she immediately knows that Bob sent her one of  $\{|\uparrow, x\rangle, |\uparrow, y\rangle, |\downarrow, z\rangle\}$ . If Bob now tells her the axis along which he measured, Alice is able to tell which result he got. This leads to a geometric way of solving the problem by visualizing the  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$  and  $|\phi_j\rangle$  states as vectors in  $\mathcal{H}_{AB}$ . Alice projects the vector she gets from Bob onto her basis  $\{|\phi_j\rangle\}$  and gets a result  $\lambda_i$ . But then she has only enough information if each of her basis vectors is orthogonal to three of the six possible vectors Bob can send. Only then can she exclude these three and only the three other vectors can lead to the specific  $\lambda_i$  she measured. If she chooses the states orthogonal to each of her basis vectors to be one of each of the three pairs

<sup>2</sup>Even then they will only be isomorphic, as  $|\uparrow\rangle_x \in \mathcal{H}_B$  and  $|\uparrow, x\rangle \in \mathcal{H}_{AB}$  with  $\mathcal{H}_A$  trivial.

$\{(|\uparrow, x\rangle, |\downarrow, x\rangle), (|\uparrow, y\rangle, |\downarrow, y\rangle), (|\uparrow, z\rangle, |\downarrow, z\rangle)\}$  the problem is now solved. If Bob tells her the axis along which he measured, Alice knows the state he obtained.

This geometric point of view yields the defining equations for the basis  $\{|\phi_j\rangle\}$  for a given  $|\psi\rangle_{AB}$ . As mentioned above, Alice cannot send an unentangled qubit to solve the problem. This suggests preparing  $|\psi\rangle_{AB}$  as an entangled state of two qubits yielding a four-dimensional Hilbert space  $\mathcal{H}_{AB}$ . The entanglement of the states is necessary as Bob will measure only in a two-dimensional subspace of  $\mathcal{H}_{AB}$ . This means that, if an unentangled state was prepared, the Hilbertspace  $\mathcal{H}_A$  would not be used at all and could as well be omitted. In fact the four-dimensional space of two qubits turns out to be necessary and sufficient to solve this case of the problem.

If Alice makes a guess and prepares  $|\psi\rangle_{AB}$  to be the singlet state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_z - |\downarrow\uparrow\rangle_z) \quad (3)$$

which - up to a global phase factor - looks the same in all bases. Here, the short hand notation  $|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle$  is used. Then  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$  can be specified to be proportional to  $\{|\uparrow\downarrow\rangle_x, |\downarrow\uparrow\rangle_x, |\uparrow\downarrow\rangle_y, |\downarrow\uparrow\rangle_y, |\uparrow\downarrow\rangle_z, |\downarrow\uparrow\rangle_z\}$ . Now Alice wants to have the following look-up table:

|             | $x$          | $y$          | $z$          |
|-------------|--------------|--------------|--------------|
| $\lambda_1$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $\lambda_2$ | $\uparrow$   | $\uparrow$   | $\downarrow$ |
| $\lambda_3$ | $\downarrow$ | $\uparrow$   | $\uparrow$   |
| $\lambda_4$ | $\uparrow$   | $\downarrow$ | $\uparrow$   |

That is, she wants

$$\begin{aligned}
|\phi_1\rangle &\perp \{|\uparrow, x\rangle, |\uparrow, y\rangle, |\uparrow, z\rangle\}, \\
|\phi_2\rangle &\perp \{|\downarrow, x\rangle, |\downarrow, y\rangle, |\uparrow, z\rangle\}, \\
|\phi_3\rangle &\perp \{|\uparrow, x\rangle, |\downarrow, y\rangle, |\downarrow, z\rangle\}, \\
|\phi_4\rangle &\perp \{|\downarrow, x\rangle, |\uparrow, y\rangle, |\downarrow, z\rangle\}.
\end{aligned} \quad (4)$$

and this choice of orthogonality conditions (4) leads to the four orthonormal vectors

$$\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}} \cdot |\uparrow\downarrow\rangle + \frac{1}{2} \cdot \left[ |\downarrow\downarrow\rangle \cdot e^{\frac{i\pi}{4}} - |\uparrow\uparrow\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}} \cdot |\uparrow\downarrow\rangle - \frac{1}{2} \cdot \left[ |\downarrow\downarrow\rangle \cdot e^{\frac{i\pi}{4}} - |\uparrow\uparrow\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}} \cdot |\downarrow\uparrow\rangle + \frac{1}{2} \cdot \left[ |\downarrow\downarrow\rangle \cdot e^{\frac{-i\pi}{4}} - |\uparrow\uparrow\rangle \cdot e^{\frac{i\pi}{4}} \right]
\end{aligned}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{2}} \cdot |\downarrow\uparrow\rangle - \frac{1}{2} \cdot \left[ |\downarrow\downarrow\rangle \cdot e^{\frac{-i\pi}{4}} - |\uparrow\uparrow\rangle \cdot e^{\frac{i\pi}{4}} \right] \quad (5)$$

all expressed in the z-basis. (5) together with the state (3) and the above look-up table solves the problem. (5) is very similar to the result that is given in [1] and which is included in the appendix. In fact all possible bases can be obtained from one by a unitary transformation, as we shall see.

Now one could ask whether Alice really needs to have four distinct eigenvalues  $\lambda_j$  and therefore a big look-up table, or whether the number of eigenvalues can be reduced. However, the reduction of distinct eigenvalues turns out to be impossible, whatever Hilbertspace  $\mathcal{H}_A$  is used. To prove this, we assume  $\mathcal{H}_A$  to be  $k$ -dimensional with basis  $\{|i\rangle : 1 \leq i \leq k\}$ . Then the state Alice prepares can be written as

$$\begin{aligned} |\psi\rangle_{AB} &= \sum_{j=1}^k \sum_{\alpha \in \{\uparrow, \downarrow\}} a_{j,\alpha} |j\rangle \otimes |\alpha\rangle_z \\ &= \sum_j a_{j,\uparrow} |j\rangle \otimes |\uparrow\rangle_z + \sum_j a_{j,\downarrow} |j\rangle \otimes |\downarrow\rangle_z \\ &= \frac{1}{\sqrt{2}} \cdot \left( \sum_j a_{j,\uparrow} |j\rangle \otimes (|\uparrow\rangle_x + |\downarrow\rangle_x) + \sum_j a_{j,\downarrow} |j\rangle \otimes (|\uparrow\rangle_x - |\downarrow\rangle_x) \right) \\ &= \frac{1}{\sqrt{2}} \cdot \left( \sum_j a_{j,\uparrow} |j\rangle \otimes (|\uparrow\rangle_y + |\downarrow\rangle_y) + \sum_j a_{j,\downarrow} |j\rangle \otimes (-i)(|\uparrow\rangle_y - |\downarrow\rangle_y) \right). \end{aligned} \quad (6)$$

Bob's measurement leads to either of

$$|\phi_{+1}(\mathbf{n}_z)\rangle = \sum_i a_{i,\uparrow} |i\rangle \otimes |\uparrow\rangle_z \quad (7)$$

$$|\phi_{-1}(\mathbf{n}_z)\rangle = \sum_i a_{i,\downarrow} |i\rangle \otimes |\downarrow\rangle_z \quad (8)$$

$$\begin{aligned} |\phi_{+1}(\mathbf{n}_x)\rangle &= \left( \sum_i a_{i,\uparrow} |i\rangle + \sum_i a_{i,\downarrow} |i\rangle \right) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle_z + |\downarrow\rangle_z) \\ &= \frac{1}{\sqrt{2}} \left( \sum_i a_{i,\uparrow} |i, \uparrow\rangle_z + \sum_i a_{i,\uparrow} |i, \downarrow\rangle_z + \sum_i a_{i,\downarrow} |i, \uparrow\rangle_z + \sum_i a_{i,\downarrow} |i, \downarrow\rangle_z \right) \end{aligned} \quad (9)$$

$$\begin{aligned} |\phi_{-1}(\mathbf{n}_x)\rangle &= \left( \sum_i a_{i,\uparrow} |i\rangle - \sum_i a_{i,\downarrow} |i\rangle \right) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle_z - |\downarrow\rangle_z) \\ &= \frac{1}{\sqrt{2}} \left( \sum_i a_{i,\uparrow} |i, \uparrow\rangle_z - \sum_i a_{i,\uparrow} |i, \downarrow\rangle_z - \sum_i a_{i,\downarrow} |i, \uparrow\rangle_z + \sum_i a_{i,\downarrow} |i, \downarrow\rangle_z \right) \end{aligned} \quad (10)$$

$$\begin{aligned} |\phi_{+1}(\mathbf{n}_y)\rangle &= \left( \sum_i a_{i,\uparrow} |i\rangle - i \sum_i a_{i,\downarrow} |i\rangle \right) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle_z + i|\downarrow\rangle_z) \\ &= \frac{1}{\sqrt{2}} \left( \sum_i a_{i,\uparrow} |i, \uparrow\rangle_z + i \sum_i a_{i,\uparrow} |i, \downarrow\rangle_z - i \sum_i a_{i,\downarrow} |i, \uparrow\rangle_z + \sum_i a_{i,\downarrow} |i, \downarrow\rangle_z \right) \end{aligned} \quad (11)$$

$$\begin{aligned} |\phi_{-1}(\mathbf{n}_y)\rangle &= \left( \sum_i a_{i,\uparrow} |i\rangle + i \sum_i a_{i,\downarrow} |i\rangle \right) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle_z - i|\downarrow\rangle_z) \\ &= \frac{1}{\sqrt{2}} \left( \sum_i a_{i,\uparrow} |i, \uparrow\rangle_z - i \sum_i a_{i,\uparrow} |i, \downarrow\rangle_z + i \sum_i a_{i,\downarrow} |i, \uparrow\rangle_z + \sum_i a_{i,\downarrow} |i, \downarrow\rangle_z \right). \end{aligned} \quad (12)$$

We see that (12) =  $\sqrt{2} \cdot (7) + \sqrt{2} \cdot (8) - (11)$  and (10) =  $\sqrt{2} \cdot (7) + \sqrt{2} \cdot (8) - (9)$ . So we showed that the six results that

can be obtained by Bob's measurement are not linearly independent but lie in a four-dimensional subspace of  $\mathcal{H}_{AB}$ . This however tells us that the number of distinct  $\lambda_j$  has to be four. This can be easily understood in the geometric picture adopted above. Assume there are only three distinct  $\lambda_j$ . However big  $\mathcal{H}_{AB}$  is, the subspace that contains the states (7)-(12) is four-dimensional and thus  $\mathcal{H}_{AB}$  can be assumed to be four-dimensional as well. To ensure that Alice gets the look-up table she wants, two of the four basis vectors  $|\phi_j\rangle$  have to be orthogonal to three states  $\{|\phi_{\eta_x}(\mathbf{n}_x)\rangle, |\phi_{\eta_y}(\mathbf{n}_y)\rangle, |\phi_{\eta_z}(\mathbf{n}_z)\rangle\}$  with  $\eta_l = 1$  or  $-1$ . The latter however are linearly independent and hence span a three-dimensional space. Therefore Alice cannot find a two-dimensional subspace orthogonal to all three. This proves that Alice really needs four distinct values  $\lambda_j$  and the size of the look-up table cannot be reduced.

We also note that as soon as  $m \geq 3$  we no longer have a mixture of prediction and retrodiction. Alice can only retrodict the results Bob obtained after having projected onto her final basis. If she prepared  $|\psi\rangle_{AB}$  to be an eigenstate of one of the  $\mathbf{1} \otimes \hat{\sigma} \cdot \mathbf{n}_l$  she would have to find out the result Bob obtained by measuring along one of the two remaining axes by a single measurement, which is impossible. On the other hand Alice cannot send an eigenstate of two distinct spin operators even if entangled states are used, as we have  $[\mathbf{1} \otimes \hat{\sigma}_i, \mathbf{1} \otimes \hat{\sigma}_j] = 2i\varepsilon_{ijk} \mathbf{1} \otimes \hat{\sigma}_k$ , and thus the measurement operators still do not commute.

### 3 Non-orthogonal axes

After having looked at the orthogonal case from a geometric point of view, we will now tackle the case in which Bob can measure along non-orthogonal axes and see that the number of axes can even be increased. For that purpose we shall take a more algebraic route, following [2].

In the previous case in section 2, Alice's basis states  $|\phi_j\rangle$ , and therefore the operator  $\widehat{\mathcal{M}}$ , could be derived from the condition that each of Alice's basis states  $|\phi_i\rangle$  is orthogonal to three of Bob's possible results  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$ . Alice just guessed that  $|\psi\rangle_{AB}$  might either be (3) or (2), hence calculated  $|\phi_{\eta_l}(\mathbf{n}_l)\rangle$  and via (4) found a basis for  $\widehat{\mathcal{M}}$ . However, to solve the problem in general, both  $|\psi\rangle_{AB}$  and  $\widehat{\mathcal{M}}$  should be calculated, i.e.  $|\psi\rangle_{AB}$  should not be found by trial and error. To attack this problem, we first note that Alice's initial state can be written as

$$|\psi\rangle_{AB} = \sum_{j=1}^{2k} b_j |\phi_j\rangle, \quad (13)$$

where  $\{|\phi_j\rangle\}$  is a basis of  $\mathcal{H}_{AB}$ . Without loss of generality the  $b_j$  can be assumed to be real. Then the problem reduces to finding the basis states  $|\phi_j\rangle$  together with the values  $b_j$ , as these give both  $\widehat{\mathcal{M}}$  and  $|\psi\rangle_{AB}$ . To find these we look at the conditions they have to satisfy, to ensure Alice can retrodict Bob's results. First of all  $|\psi\rangle_{AB}$  should be normalised, which gives:

$$\sum_j b_j^2 = 1. \quad (14)$$

A second condition is that, in order to enable Alice to retrodict Bob's results  $\eta_l$  deterministically, the states  $|\phi_{+1}(\mathbf{n}_l)\rangle$  and  $|\phi_{-1}(\mathbf{n}_l)\rangle$  must lie in the span of disjoint subsets of  $\{|\phi_j\rangle\}$ . Otherwise the result  $\lambda_i$ , which Alice gets if she measures using  $\mathcal{M}$ , would not tell her  $\eta_l$ . As this must be true for all axes  $l$  along which Bob can measure, there are  $m$  pairs of disjoint subsets. Therefore, let us introduce  $2m$  sets of indices  $S_{\eta_l}(\mathbf{n}_l)$  that indicate which basis vectors span the vectors

$$|\phi_{\eta_l}(\mathbf{n}_l)\rangle = \sum_{j \in S_{\eta_l}(\mathbf{n}_l)} b_j |\phi_j\rangle. \quad (15)$$

Here the  $b_j$  are the same as in (13), which can be seen from

$$|\psi\rangle_{AB} = |\phi_+(\mathbf{n}_l)\rangle + |\phi_-(\mathbf{n}_l)\rangle. \quad (16)$$

This equation only states that Bob always gets either ‘up’ or ‘down’. So the  $S_{\eta_l}(\mathbf{n}_l)$  provide a look-up table for Alice. In the case she measures  $\lambda_i$  corresponding to  $|\phi_i\rangle$ , Alice checks whether  $i \in S_{+1}(\mathbf{n}_l)$  or  $i \in S_{-1}(\mathbf{n}_l)$  which tells her which result Bob got if he measured along  $\mathbf{n}_l$ .

For further convenience it is helpful to introduce a sign function for the partitions  $S_{\eta_l}(\mathbf{n}_l)$  in the following way:

$$\varepsilon_j^{(l)} = \begin{cases} +1 & \text{for } j \in S_{+1}(\mathbf{n}_l) \\ -1 & \text{for } j \in S_{-1}(\mathbf{n}_l) \end{cases} \quad (17)$$

For example, if  $|\phi_{+1}(\mathbf{n}_l)\rangle = b_1|\phi_1\rangle + b_3|\phi_3\rangle + b_6|\phi_6\rangle$ ,  $\varepsilon_1^{(l)}$ ,  $\varepsilon_3^{(l)}$  and  $\varepsilon_6^{(l)}$  give +1, the others give -1.

Now some algebraic manipulations have to be done to obtain sufficient information for Alice to construct her states. The basic task is to find some constraints for the look-up table that Alice chooses. Once a “good” table is chosen, the basis states can be constructed. From (16) we get (recalling (1)):

$$\mathbf{1} \otimes (\hat{\sigma} \cdot \mathbf{n}_l) |\psi\rangle_{AB} = \sum_j \varepsilon_j^{(l)} b_j |\phi_j\rangle \quad \text{for } 1 \leq l \leq m. \quad (18)$$

Furthermore the Pauli-matrices satisfy<sup>3</sup>:

$$((\widehat{\mathbf{1} \otimes \sigma}) \cdot \mathbf{n}_a) \cdot ((\widehat{\mathbf{1} \otimes \sigma}) \cdot \mathbf{n}_b) = \mathbf{n}_a \cdot \mathbf{n}_b + i(\mathbf{n}_a \times \mathbf{n}_b) \cdot \widehat{\mathbf{1} \otimes \sigma}. \quad (19)$$

The expectation value of (19) yields:

$${}_{AB} \langle \psi | \mathbf{n}_a \cdot \mathbf{n}_b | \psi \rangle_{AB} + i {}_{AB} \langle \psi | (\mathbf{n}_a \times \mathbf{n}_b) \cdot \widehat{\mathbf{1} \otimes \sigma} | \psi \rangle_{AB} = \sum_j \varepsilon_j^{(a)} \varepsilon_j^{(b)} b_j^2 \quad (20)$$

and therefore the two equations,

$$0 = {}_{AB} \langle \psi | (\mathbf{n}_l \cdot (\widehat{\mathbf{1} \otimes \sigma})) | \psi \rangle_{AB} = \sum_j \varepsilon_j^{(l)} b_j \quad (21)$$

and

$$\mathbf{n}_l \cdot \mathbf{n}_j = \sum_s \varepsilon_s^{(l)} \varepsilon_s^{(j)} b_s^2, \quad (22)$$

for the real and imaginary components. From (21) we obtain:

$$\sum_{j \in S_{\eta_l}} b_j^2 = \frac{1}{2}, \quad (23)$$

which tells us that the square of the coefficients of the basis states  $|\phi_j\rangle$  have to add to  $\frac{1}{2}$  for each disjoint subset. Finally we note that in the case in which  $m > 3$ , of course three axes are sufficient to span all the others:

$$\mathbf{n}_{k+3} = \sum_{l=1}^3 c_l^{(k)} \mathbf{n}_l \quad \text{for } 1 \leq k \leq m-3, \quad (24)$$

---

<sup>3</sup>Note that  $(\widehat{\mathbf{1} \otimes \sigma}) \cdot \mathbf{n}_a = \mathbf{1} \otimes (\hat{\sigma} \cdot \mathbf{n}_a)$ .



and thus from (18):

$$\varepsilon_s^{(k+3)} = \sum_{l=1}^3 c_l^{(k)} \varepsilon_s^{(l)} \quad \text{for} \quad 1 \leq s \leq 2k. \quad (25)$$

It turns out that these relations enable Alice to calculate both the  $b_j$  and the  $|\phi_j\rangle$ . It was shown in [2] that as a necessary and sufficient condition for the construction of a basis we need (i)  $\varepsilon_l^{(l)}$  and  $b_j$  that solve (14), (21) and (22); and (ii) if  $m \geq 3$ , there exist numbers  $c_l^{(k)}$   $l \leq l \leq 3$  and  $l \leq k \leq m-3$  such that (25) holds. Furthermore it was proved that no solutions exist for  $m > 4$  and that if  $m = 4$  the directions have to satisfy  $\sum_{l=1}^4 \mathbf{n}_l = 0$ . But all this only means that the look-up table has to satisfy certain conditions. This, however, was anticipated already as not any arbitrary look-up table should lead to a solution.

However, once Alice chooses a suitable look-up table, the basis states can be derived. The orientation of Bob's axes (22), together with Alice condition (23), lead to a unique set of components  $b_j$ . The crucial equation (18) now tells us the action of Bob's measurement on  $|\psi\rangle_{AB}$ . These can be inverted to obtain  $\{|\phi_j\rangle\}$ . The details of this process can best be understood by looking at an example.

In the following we will explicitly construct the basis for the case  $m=4$ , using this method. The non-orthogonal  $m = 3$  case can be solved equivalently and a possible basis is given in the appendix.

The problem is solved using a six-dimensional Hilbert space  $\mathcal{H}_{AB}$ . A basis of this space is denoted by  $\{|2, 1\rangle, |2, -1\rangle, \dots, |0, -1\rangle$  where  $|\rho, \zeta\rangle = |\rho\rangle \otimes |\zeta\rangle$  with  $\rho = 1, 2, 3$  and  $\zeta = \pm 1$  is the tensor product of a *qutrit* and a qubit.

We use the partition provided in [2], namely

$$S_+(\mathbf{n}_1) = \{1, 2, 3\},$$

$$S_+(\mathbf{n}_2) = \{1, 5, 6\},$$

$$S_+(\mathbf{n}_3) = \{3, 4, 6\}. \quad (26)$$

which satisfy the necessary conditions. From (25) we get  $\varepsilon_1^{(4)} = \varepsilon_3^{(4)} = \varepsilon_6^{(4)} = -1$ ;  $\varepsilon_2^{(4)} = \varepsilon_4^{(4)} = \varepsilon_5^{(4)} = 1$ . This is nothing but the following look-up table for Alice:

|             | $\mathbf{n}_1$ | $\mathbf{n}_2$ | $\mathbf{n}_3$ | $\mathbf{n}_4$ |
|-------------|----------------|----------------|----------------|----------------|
| $\lambda_1$ | $\uparrow$     | $\uparrow$     | $\downarrow$   | $\downarrow$   |
| $\lambda_2$ | $\uparrow$     | $\downarrow$   | $\downarrow$   | $\uparrow$     |
| $\lambda_3$ | $\uparrow$     | $\downarrow$   | $\uparrow$     | $\downarrow$   |
| $\lambda_4$ | $\downarrow$   | $\downarrow$   | $\uparrow$     | $\uparrow$     |
| $\lambda_5$ | $\downarrow$   | $\uparrow$     | $\downarrow$   | $\uparrow$     |
| $\lambda_6$ | $\downarrow$   | $\uparrow$     | $\uparrow$     | $\downarrow$   |

From (23) this table gives immediatly

$$b_1^2 + b_2^2 + b_3^2 = b_4^2 + b_5^2 + b_6^2 = b_1^2 + b_5^2 + b_6^2 = b_3^2 + b_4^2 + b_6^2 = \frac{1}{2}. \quad (27)$$

These equations can be solved in terms of the two free parameters  $b_5$  and  $b_6$ :

$$b_1^2 = b_4^2 = \frac{1}{2} - b_5^2 - b_6^2; \quad b_2^2 = b_6^2; \quad b_3^2 = b_5^2. \quad (28)$$

(22) yields:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 - 4b_5^2 - 4b_6^2; \quad \mathbf{n}_2 \cdot \mathbf{n}_3 = 4b_6^2 - 1; \quad \mathbf{n}_3 \cdot \mathbf{n}_1 = 4b_5^2 - 1. \quad (29)$$

Therefore the two parameters  $b_5$ ,  $b_6$  specify the mutual orientation of the axes. These are all the preliminaries we need to construct our basis. To specify the axes we make the symmetric choice:

$$b_j = \frac{1}{\sqrt{6}} \quad \forall j.$$

The crucial step is that (18) tells us the action of  $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$  on  $|\psi\rangle_{AB}$ , giving four orthonormal states:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{6}}(|\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle + |\phi_5\rangle + |\phi_6\rangle) \quad (30)$$

$$(\mathbf{1} \otimes \hat{\sigma}_x)|\psi\rangle_{AB} = \frac{1}{\sqrt{6}}(|\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle - |\phi_4\rangle - |\phi_5\rangle - |\phi_6\rangle) \quad (31)$$

$$(\mathbf{1} \otimes \hat{\sigma}_y)|\psi\rangle_{AB} = \frac{1}{\sqrt{12}}(2 \cdot |\phi_1\rangle - |\phi_2\rangle - |\phi_3\rangle - 2 \cdot |\phi_4\rangle + |\phi_5\rangle + |\phi_6\rangle) \quad (32)$$

$$(\mathbf{1} \otimes \hat{\sigma}_z)|\psi\rangle_{AB} = \frac{1}{2}(-|\phi_2\rangle + |\phi_3\rangle - |\phi_5\rangle + |\phi_6\rangle) \quad (33)$$

These are orthonormal and can be extended to a basis by

$$|\chi_1\rangle = \frac{1}{2} \cdot (-|\phi_1\rangle + |\phi_2\rangle - |\phi_4\rangle + |\phi_6\rangle) \quad (34)$$

and

$$|\chi_2\rangle = \frac{1}{12} \cdot (-|\phi_1\rangle - |\phi_2\rangle + 2 \cdot |\phi_3\rangle - |\phi_4\rangle + 2 \cdot |\phi_5\rangle - |\phi_6\rangle). \quad (35)$$

These equations have to be inverted to get explicit expressions for the basis states  $|\phi_j\rangle$ . In order to express them in terms of the basis  $\{|2, 1\rangle, |2, -1\rangle, \dots, |0, -1\rangle$  of the six-dimensional vector space  $\mathcal{H}_{AB}$ , we note that  $\frac{1}{\sqrt{2}} \cdot ((\mathbf{1} \otimes \hat{\sigma}_x) + i\zeta(\mathbf{1} \otimes \hat{\sigma}_y))|\psi\rangle_{AB}$  is an eigenstate of  $(\mathbf{1} \otimes \hat{\sigma}_z)$  with eigenvalue  $\zeta = \pm 1$ , as well as  $\frac{1}{\sqrt{2}} \cdot ((\mathbf{1} \otimes \mathbf{1} + \zeta(\mathbf{1} \otimes \hat{\sigma}_y))|\psi\rangle_{AB}$  and  $(|\chi_1\rangle \cdot \delta_{1,\zeta} + |\chi_2\rangle \cdot \delta_{-1,\zeta})$ , with the usual Kronecker symbol  $\delta_{ij}$ . Hence these states form a basis of the three-dimensional eigenspace of the operator  $\mathbf{1} \otimes \hat{\sigma}_z$  with eigenvalue  $\zeta$ . However  $|2, \zeta\rangle, |1, \zeta\rangle, |0, \zeta\rangle$ , is also a basis of this subspace, therefore there has to be a unitary transformation, such that

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \cdot ((\mathbf{1} \otimes \hat{\sigma}_x) + i\zeta(\mathbf{1} \otimes \hat{\sigma}_y))|\psi\rangle_{AB} \\ \frac{1}{\sqrt{2}} \cdot (\mathbf{1} \otimes \mathbf{1} + \zeta(\mathbf{1} \otimes \hat{\sigma}_y))|\psi\rangle_{AB} \\ |\chi_1\rangle \cdot \delta_{1,\zeta} + |\chi_2\rangle \cdot \delta_{-1,\zeta} \end{pmatrix} = e^{i\hat{\theta}_\zeta \cdot \hat{\tau} + i\Lambda_\zeta} \cdot \begin{pmatrix} |2, \zeta\rangle \\ |1, \zeta\rangle \\ |0, \zeta\rangle \end{pmatrix}. \quad (36)$$

Here  $\hat{\theta}_\pm$  are arbitrary eight-vectors, and  $\hat{\tau}$  is a vector consisting of the eight generators of  $SU(3)$ ;  $\Lambda_\pm$  are arbitrary numbers. Thus, the most general solution is characterised by a set of 24 numbers  $(\boldsymbol{\theta}_\pm, \Lambda_\pm, \{\lambda_j\})$ . Equation (36) allows us to express the basis states  $|\phi_j\rangle$  in terms of the computational basis of  $\mathcal{H}_{AB}$ . Without loss of generality, we choose  $\boldsymbol{\theta}_\zeta=0$  and  $\Lambda_\zeta=0$  for both  $\zeta$  and finally get:

$$\begin{aligned}
|\phi_1\rangle &= \left(\frac{1}{\sqrt{12}} - \frac{i}{\sqrt{6}}\right)|2, \uparrow\rangle + \frac{1}{\sqrt{12}}|1, \uparrow\rangle - \frac{1}{2}|0, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} + \frac{i}{\sqrt{6}}\right)|2, \downarrow\rangle + \frac{1}{\sqrt{12}}|1, \downarrow\rangle - \frac{1}{\sqrt{12}}|0, \downarrow\rangle \\
|\phi_2\rangle &= \left(\frac{-1}{\sqrt{12}} + \frac{i}{\sqrt{6}}\right)|2, \uparrow\rangle + \frac{1}{\sqrt{12}}|1, \uparrow\rangle - \frac{1}{2}|0, \uparrow\rangle + \left(\frac{-1}{\sqrt{12}} - \frac{i}{\sqrt{6}}\right)|2, \downarrow\rangle + \frac{1}{\sqrt{12}}|1, \downarrow\rangle - \frac{1}{\sqrt{12}}|0, \downarrow\rangle \\
|\phi_3\rangle &= \left(\frac{1}{\sqrt{12}} + \frac{i}{\sqrt{24}}\right)|2, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{8}}\right)|1, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} - \frac{i}{\sqrt{24}}\right)|2, \downarrow\rangle + \left(\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{8}}\right)|1, \downarrow\rangle + \frac{1}{\sqrt{3}}|0, \downarrow\rangle \\
|\phi_4\rangle &= \left(\frac{-1}{\sqrt{12}} - \frac{i}{\sqrt{24}}\right)|2, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{8}}\right)|1, \uparrow\rangle + \left(\frac{-1}{\sqrt{12}} + \frac{i}{\sqrt{24}}\right)|2, \downarrow\rangle + \left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{8}}\right)|1, \downarrow\rangle + \frac{1}{\sqrt{3}}|0, \downarrow\rangle \\
|\phi_5\rangle &= \left(\frac{-1}{\sqrt{12}} - \frac{i}{\sqrt{24}}\right)|2, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{8}}\right)|1, \uparrow\rangle + \frac{1}{2}|0, \uparrow\rangle + \left(\frac{-1}{\sqrt{12}} + \frac{i}{\sqrt{24}}\right)|2, \downarrow\rangle + \left(\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{8}}\right)|1, \downarrow\rangle - \frac{1}{\sqrt{12}}|0, \downarrow\rangle \\
|\phi_6\rangle &= \left(\frac{1}{\sqrt{12}} + \frac{i}{\sqrt{24}}\right)|2, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{8}}\right)|1, \uparrow\rangle + \frac{1}{2}|0, \uparrow\rangle + \left(\frac{1}{\sqrt{12}} - \frac{i}{\sqrt{24}}\right)|2, \downarrow\rangle + \left(\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{12}}\right)|1, \downarrow\rangle - \frac{1}{\sqrt{12}}|0, \downarrow\rangle,
\end{aligned}$$

all expressed in the z-basis.

So we have explicitly constructed a basis for a special symmetric orientation of our axes. This technique can be applied to all possible orientations that satisfy the condition  $\sum_{l=1}^4 \mathbf{n}_l = 0$ .

## 4 The Quantum Network

After having explicitly constructed some measurement bases, we now want to know how Alice's measurements can actually be performed. Thus, we want to construct a device that prepares Alice's initial state, and the operator  $\widehat{\mathcal{M}}$  that projects onto one of the bases that were just determined.

In principle, this can be achieved by constructing a quantum network which is a kind of building plan for Alice's devices. For that purpose we regard the Hilbert space  $\mathcal{H}_{AB}$  as a subspace of the  $2^N$ -dimensional space  $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_B \otimes \dots \otimes \mathcal{H}_B$ . The basis of this space is taken to be the tensor product of the qubit basis of  $\mathcal{H}_B$ , i.e. the states  $|\uparrow\uparrow\dots\uparrow\rangle, |\uparrow\uparrow\dots\downarrow\rangle, \dots$ <sup>4</sup>. In this space of qubits we can now perform certain operations and in that way try to construct the whole experimental device out of a few basic transformations which we call quantum gates. As we demand reversibility, quantum gates have to be unitary. (We could also say, that as we have *quantum* gates which obey the rules of quantum mechanics, we have unitary transformations and therefore reversibility). It was shown in [4] that there is a set of universal quantum gates from which every evolution of a quantum system can be approximated with arbitrary precision. A set of universal gates can be taken to be the *Phase Shift*  $P(\phi)$ , defined by

$$\begin{aligned}
|0\rangle &\mapsto |0\rangle \\
|1\rangle &\mapsto e^{i\phi}|1\rangle,
\end{aligned} \tag{37}$$

the *Controlled Not* (*CNOT*),

$$\begin{aligned}
|00\rangle &\mapsto |00\rangle \\
|01\rangle &\mapsto |01\rangle \\
|10\rangle &\mapsto |11\rangle \\
|11\rangle &\mapsto |10\rangle,
\end{aligned} \tag{38}$$

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<sup>4</sup>It is convenient to use  $|0\rangle := |\uparrow\rangle$  and  $|1\rangle := |\downarrow\rangle$  in this section.

and the *Hadamard gate* ( $H$ ),

$$\begin{aligned} |0\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |1\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \end{aligned} \quad (39)$$

If we use the representation,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \quad (40)$$

the basic transformations can be written as:

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (41)$$

Note that these gates only act on one or two qubits. The fact that they are nevertheless universal is very important as these gates can, in principle, be built and therefore even complicated calculations can be done without using gates that act on many qubits at once.

These gates can also be visualized in an elegant way. We denote the *Phase Shift* by

$$|x\rangle \xrightarrow{\phi} e^{ix\phi} |x\rangle, \quad (42)$$

the *Hadamard gate* by

$$|x\rangle \xrightarrow{H} \frac{(-1)^x |x\rangle + |1-x\rangle}{\sqrt{2}} \quad (43)$$

and the *Controlled Not* by

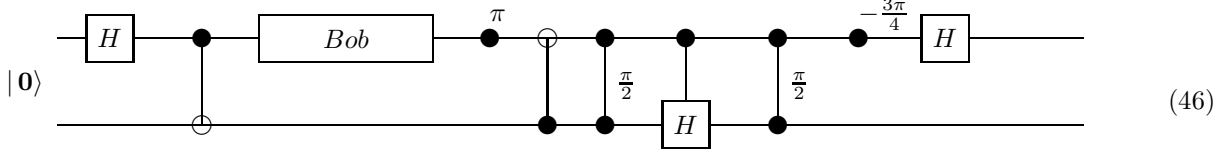
$$\begin{array}{c} |x\rangle \text{---} \bullet \text{---} \\ |y\rangle \text{---} \circ \text{---} \end{array} \quad |x\rangle |y+x\rangle, \quad (44)$$

where on the left hand side of the diagram the input state is given, the output on the right, and  $x, y \in \{0, 1\}$ .

Finally we note that the *Controlled Phase Shift*  $CP(\phi)$  can be built out of the above states. It is denoted by

$$CP(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \quad \begin{array}{c} |x\rangle \text{---} \bullet \text{---} \\ |y\rangle \text{---} \bullet \text{---} \end{array} \quad e^{ixy\phi} |x\rangle |y\rangle. \quad (45)$$

After having set up the basic framework we will present networks that prepare Alice's initial state and project onto the basis  $|\phi_j\rangle$ . Let us first look at the case in which the symmetric state (2) is prepared, which is projected onto (54). In this case we have the following network:



We use the convention that the original state is described by  $|\mathbf{0}\rangle = |0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle \in (\mathcal{H}_B)^N = \mathcal{H}$ . Then the first part of the network gives:

$$|\mathbf{0}\rangle = |0\rangle \otimes |0\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{C_{N_2}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (47)$$

Here the indices denote the qubit on which the transformation is performed. To prove that the second part projects onto the basis (54), we have to show that the set of basis vectors in (54) is mapped bijectively onto  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ . For that purpose we introduce yet another gate, the *Controlled Unitary gate (CU)*:

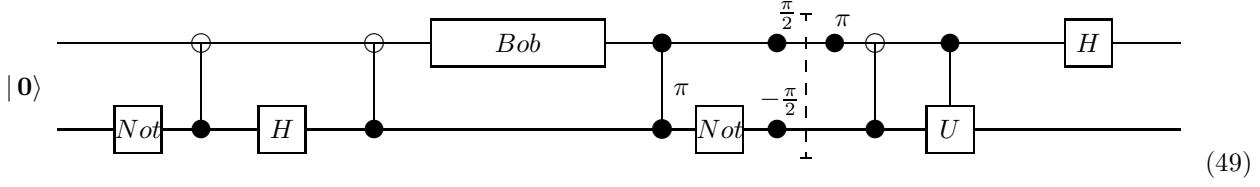
$$CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{2}} & i \cdot \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{2}} \\ 0 & 0 & i \cdot \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{2}} & \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{2}} \end{pmatrix}. \quad (48)$$

The *Controlled Hadamard (CH)* can be built directly.

Now we can easily check:

$$\begin{aligned} |\phi_{1,2}\rangle &= \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} + |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ P(\pi)_1 &\xrightarrow{P(\pi)_1} \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} - |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ C_{N_1} &\xrightarrow{C_{N_1}} \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ |11\rangle \cdot e^{\frac{i\pi}{4}} - |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ C_{U_2} &\xrightarrow{C_{U_2}} \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ \frac{1}{\sqrt{2}}(|11\rangle + i|10\rangle) \cdot e^{\frac{-i\pi}{2}} - \frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle) \cdot e^{-i\pi} \right] = \frac{1}{\sqrt{2}}|00\rangle \pm |10\rangle \\ H_1 &\xrightarrow{H_1} \begin{cases} |00\rangle & \text{for } |\phi_1\rangle \\ |10\rangle & \text{for } |\phi_2\rangle \end{cases} \\ |\phi_{3,4}\rangle &= \frac{1}{\sqrt{2}}|11\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} + |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ P(\pi) &\xrightarrow{P(\pi)} \frac{1}{\sqrt{2}}|11\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} - |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ C_{N_1} &\xrightarrow{C_{N_1}} \frac{1}{\sqrt{2}}|01\rangle \pm \frac{1}{2} \left[ |11\rangle \cdot e^{\frac{i\pi}{4}} - |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ C_{U_2} &\xrightarrow{C_{U_2}} \frac{1}{\sqrt{2}}|01\rangle \pm \frac{1}{2} \left[ \frac{1}{\sqrt{2}}(|11\rangle + i|10\rangle) \cdot e^{-i\frac{\pi}{2}} - \frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle) \cdot e^{-i\pi} \right] = \frac{1}{\sqrt{2}}|00\rangle \pm |10\rangle \\ H_1 &\xrightarrow{H_1} \begin{cases} |01\rangle & \text{for } |\phi_3\rangle \\ |11\rangle & \text{for } |\phi_4\rangle \end{cases} \end{aligned}$$

If Alice prepares the singlet state (3), the gate looks slightly more complicated:



Here the *Not gate* denotes<sup>5</sup>:

$$\text{---} \boxed{\text{Not}} \text{---} = \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \quad (50)$$

Again it can be proved that this gate prepares the correct state  $|\psi\rangle_{AB}$  and projects onto  $|\phi_j\rangle$  as given in (5). The first part gives:

$$|0\rangle = |0\rangle \otimes |0\rangle \xrightarrow{Not_2} |0\rangle \otimes |1\rangle \xrightarrow{CN_1} |1\rangle \otimes |1\rangle \xrightarrow{H_2} \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \xrightarrow{CN_1} \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle). \quad (51)$$

To prove that the second part projects onto (5), we first look at the first three steps in the diagram:

$$\begin{aligned} |\phi_{1,2}\rangle &= \frac{1}{\sqrt{2}}|01\rangle \pm \frac{1}{2} \left[ |11\rangle \cdot e^{\frac{i\pi}{4}} - |00\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{CP(\pi)} \frac{1}{\sqrt{2}}|01\rangle \pm \frac{1}{2} \left[ -|11\rangle \cdot e^{\frac{i\pi}{4}} - |00\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{Not_2} \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ -|10\rangle \cdot e^{\frac{i\pi}{4}} - |01\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{P(\frac{\pi}{2})_1, P(\frac{-\pi}{2})_2} \frac{1}{\sqrt{2}}|00\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} + |10\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ |\phi_{3,4}\rangle &= \frac{1}{\sqrt{2}}|10\rangle \pm \frac{1}{2} \left[ |11\rangle \cdot e^{\frac{i\pi}{4}} - |00\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{CP(\pi)} \frac{1}{\sqrt{2}}|10\rangle \pm \frac{1}{2} \left[ -|11\rangle \cdot e^{\frac{i\pi}{4}} - |00\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{Not_2} \frac{1}{\sqrt{2}}|11\rangle \pm \frac{1}{2} \left[ -|10\rangle \cdot e^{\frac{i\pi}{4}} - |01\rangle \cdot e^{\frac{-i\pi}{4}} \right] \\ &\xrightarrow{P(\frac{\pi}{2})_1, P(\frac{-\pi}{2})_2} \frac{1}{\sqrt{2}}|11\rangle \pm \frac{1}{2} \left[ |01\rangle \cdot e^{\frac{i\pi}{4}} + |10\rangle \cdot e^{\frac{-i\pi}{4}} \right]. \end{aligned} \quad (52)$$

But this is the basis (54). Therefore, these gates transform one basis into the other. The rest of the gate is the same as in (46) and hence the gate projects onto the correct basis.

## 5 Appendix

In the appendix we give some basic formulae as well as a different basis for the  $m=3$  orthogonal case and the basis for the  $m=3$  non-orthogonal case.

(i), Basic Transformations:

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<sup>5</sup>In fact a *Not gate* could also be built directly.

$$\begin{aligned}
|\uparrow\rangle_x &= \frac{1}{\sqrt{2}} \cdot (|\uparrow\rangle_z + |\downarrow\rangle_z) \\
|\downarrow\rangle_x &= \frac{1}{\sqrt{2}} \cdot (|\uparrow\rangle_z - |\downarrow\rangle_z) \\
|\uparrow\rangle_y &= \frac{1}{\sqrt{2}} \cdot (|\uparrow\rangle_z + i|\downarrow\rangle_z) \\
|\downarrow\rangle_y &= \frac{1}{\sqrt{2}} \cdot (|\uparrow\rangle_z - i|\downarrow\rangle_z).
\end{aligned} \tag{53}$$

(ii), In [1] the state (2) was used and the following basis was obtained:

$$\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}} |\uparrow\uparrow\rangle_z + \frac{1}{2} \left[ |\uparrow\downarrow\rangle_z \cdot e^{\frac{i\pi}{4}} + |\downarrow\uparrow\rangle_z \cdot e^{-\frac{i\pi}{4}} \right] \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}} |\uparrow\uparrow\rangle_z - \frac{1}{2} \left[ |\uparrow\downarrow\rangle_z \cdot e^{\frac{i\pi}{4}} + |\downarrow\uparrow\rangle_z \cdot e^{-\frac{i\pi}{4}} \right] \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle_z + \frac{1}{2} \left[ |\uparrow\downarrow\rangle_z \cdot e^{-\frac{i\pi}{4}} + |\downarrow\uparrow\rangle_z \cdot e^{\frac{i\pi}{4}} \right] \\
|\phi_4\rangle &= \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle_z - \frac{1}{2} \left[ |\uparrow\downarrow\rangle_z \cdot e^{-\frac{i\pi}{4}} + |\downarrow\uparrow\rangle_z \cdot e^{\frac{i\pi}{4}} \right].
\end{aligned} \tag{54}$$

This is equivalent to (5) and can be obtained by a unitary transformation.

(iii), The  $m = 3$  non-orthogonal case:

In this case Alice can choose a look-up table like:

|             | $\mathbf{n}_1$ | $\mathbf{n}_2$ | $\mathbf{n}_3$ |
|-------------|----------------|----------------|----------------|
| $\lambda_1$ | $\downarrow$   | $\uparrow$     | $\downarrow$   |
| $\lambda_2$ | $\downarrow$   | $\uparrow$     | $\uparrow$     |
| $\lambda_3$ | $\downarrow$   | $\downarrow$   | $\downarrow$   |
| $\lambda_4$ | $\downarrow$   | $\downarrow$   | $\uparrow$     |
| $\lambda_5$ | $\uparrow$     | $\uparrow$     | $\downarrow$   |
| $\lambda_6$ | $\uparrow$     | $\downarrow$   | $\uparrow$     |
| $\lambda_7$ | $\uparrow$     | $\uparrow$     | $\uparrow$     |
| $\lambda_8$ | $\uparrow$     | $\uparrow$     | $\uparrow$     |

A special choice of the orientation of axes with  $b_1^2 = b_4^2 = b_5^2 = b_6^2 = b_7^2 = b_8^2 = \frac{1}{8}$  and  $b_2 = 0$ ,  $b_3^2 = \frac{1}{8}$  leads to:

$$\begin{aligned}
|\phi_1\rangle &= |\downarrow\downarrow\downarrow\rangle \\
|\phi_2\rangle &= -\frac{1}{4}(1+i\sqrt{3})|\uparrow\uparrow\uparrow\rangle - \frac{1}{4}(1-i\sqrt{3})|\uparrow\uparrow\downarrow\rangle + \frac{1}{4}(|\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle) + \frac{\sqrt{35}}{10}|\downarrow\downarrow\uparrow\rangle - \frac{\sqrt{10}}{20}|\downarrow\downarrow\downarrow\rangle \\
|\phi_3\rangle &= +\frac{1}{4}(1+i\sqrt{3})|\uparrow\uparrow\uparrow\rangle + \frac{1}{4}(1-i\sqrt{3})|\uparrow\uparrow\downarrow\rangle + \frac{1}{4}(|\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle) + \frac{\sqrt{35}}{10}|\downarrow\downarrow\uparrow\rangle - \frac{\sqrt{10}}{20}|\downarrow\downarrow\downarrow\rangle \\
|\phi_4\rangle &= \left(-\frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{12}\right)|\uparrow\uparrow\uparrow\rangle + \left(-\frac{\sqrt{2}}{4} - i\frac{\sqrt{6}}{12}\right)|\uparrow\uparrow\downarrow\rangle + \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{3}}{6}\right)|\uparrow\downarrow\uparrow\rangle + \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{6}\right)|\uparrow\downarrow\downarrow\rangle + \\
&\quad + \frac{\sqrt{7}}{7}|\downarrow\downarrow\uparrow\rangle - \frac{1}{35}|\downarrow\downarrow\downarrow\rangle + \frac{\sqrt{5}}{10}|\downarrow\downarrow\downarrow\rangle \\
|\phi_5\rangle &= \left(\frac{1}{4} - i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\uparrow\rangle + \left(\frac{1}{4} + i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\downarrow\rangle + \left(\frac{1}{4} + \frac{\sqrt{6}}{12}\right)|\uparrow\downarrow\uparrow\rangle + \left(\frac{1}{4} - \frac{\sqrt{6}}{12}\right)|\uparrow\downarrow\downarrow\rangle + \frac{\sqrt{10}}{4}|\downarrow\downarrow\downarrow\rangle \\
|\phi_6\rangle &= \left(\frac{1}{4} - i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\uparrow\rangle + \left(\frac{1}{4} + i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\downarrow\rangle + \left(\frac{1}{4} + \frac{\sqrt{6}}{12}\right)|\uparrow\downarrow\uparrow\rangle + \left(\frac{1}{4} - \frac{\sqrt{6}}{12}\right)|\uparrow\downarrow\downarrow\rangle - \frac{\sqrt{14}}{7}|\downarrow\downarrow\uparrow\rangle - \\
&\quad - \frac{2}{35}\sqrt{35}|\downarrow\downarrow\uparrow\rangle - \frac{3}{20}\sqrt{10}|\downarrow\downarrow\downarrow\rangle \\
|\phi_7\rangle &= \left(\frac{1}{4} - i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\uparrow\rangle + \left(\frac{1}{4} + i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\downarrow\rangle + \left(\frac{1}{4} + \frac{\sqrt{6}}{6}\right)|\uparrow\downarrow\uparrow\rangle + \left(\frac{1}{4} - \frac{\sqrt{6}}{6}\right)|\uparrow\downarrow\downarrow\rangle - \frac{\sqrt{14}}{7}|\downarrow\downarrow\uparrow\rangle - \\
&\quad - \frac{3}{70}\sqrt{35}|\downarrow\downarrow\uparrow\rangle - \frac{\sqrt{10}}{20}|\downarrow\downarrow\downarrow\rangle \\
|\phi_8\rangle &= \left(-\frac{1}{4} + i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\uparrow\rangle + \left(-\frac{1}{4} - i\frac{\sqrt{3}}{12}\right)|\uparrow\uparrow\downarrow\rangle + \left(\frac{1}{4} + \frac{\sqrt{6}}{6}\right)|\uparrow\downarrow\uparrow\rangle + \left(\frac{1}{4} - \frac{\sqrt{6}}{6}\right)|\uparrow\downarrow\downarrow\rangle - \frac{\sqrt{14}}{7}|\downarrow\downarrow\uparrow\rangle - \\
&\quad - \frac{3}{70}\sqrt{35}|\downarrow\downarrow\uparrow\rangle - \frac{\sqrt{10}}{20}|\downarrow\downarrow\downarrow\rangle,
\end{aligned} \tag{55}$$

all expressed in the z-basis. As in (36), all parameters of the unitary transformation were chosen to be zero.

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